The Cagan Model with Rational Expectations

$$m_t - p_t = \gamma + \alpha\left(E_t p_{t+1} - p_t\right) + u_t$$

$$m_t = \mu_0 + \mu_1 m_{t-1} + e_t$$

$$\Downarrow$$

$$m_t = \gamma + \alpha E_t p_{t+1} + (1-\alpha) p_t + u_t$$

(1) $$\mu_0 + \mu_1 m_{t-1} + e_t = \gamma + \alpha E_t p_{t+1} + (1-\alpha) p_t + u_t$$

Conjecture a solution:

$$p_t = \phi_0 + \phi_1 m_{t-1} + \phi_2 e_t + \phi_3 u_t$$

which implies

$$p_{t+1} = \phi_0 + \phi_1 m_t + \phi_2 e_{t+1} + \phi_3 u_{t+1}$$

$$E_t p_{t+1} = \phi_0 + \phi_1 m_t$$

$$= \phi_0 + \phi_1 \left(\mu_0 + \mu_1 m_{t-1} + e_t\right)$$

Substitute into 1

$\mu_0 + \mu_1 m_{t-1} + e_t =$

$$\gamma + \alpha\left[\phi_0 + \phi_1(\mu_0 + \mu_1 m_{t-1} + e_t)\right]$$
$$+ (1-\alpha)(\phi_0 + \phi_1 m_{t-1} + \phi_2 e_t + \phi_3 u_t)$$
$$+ u_t$$

Now, equate coefficients:

(1) $\mu_0 = \gamma + \alpha\phi_0 + \alpha\phi_1\mu_0 + (1-\alpha)\phi_0$

(2) $\mu_1 = \alpha\phi_1\mu_1 + (1-\alpha)\phi_1$

(3) $1 = \alpha\phi_1 + (1-\alpha)\phi_2$

(4) $0 = (1-\alpha)\phi_3 + 1$

We have 4 equations in 4 unknowns

$$(4) \Rightarrow \quad 0 = (1-\alpha)\phi_3 + 1$$

$$-1 = (1-\alpha)\phi_3$$

$$\boxed{\phi_3 = \frac{-1}{1-\alpha}}$$

$$(2) \Rightarrow \quad \mu_1 = \alpha \phi_1 \mu_1 + (1-\alpha)\phi_1$$

$$\mu_1 = \alpha \phi_1 \mu_1 + \phi_1 - \alpha \phi_1$$

$$\mu_1 = \phi_1 (\alpha \mu_1 + 1 - \alpha)$$

$$\boxed{\phi_1 = \frac{\mu_1}{\alpha \mu_1 + 1 - \alpha}}$$

$$(3) \quad 1 = \alpha \phi_1 + (1-\alpha)\phi_2$$

$$1 = \alpha \left(\frac{\mu_1}{\alpha \mu_1 + 1 - \alpha} \right) + (1-\alpha)\phi_2$$

$$\frac{\alpha \mu_1 + 1 - \alpha}{\alpha \mu_1 + 1 - \alpha} - \frac{\alpha \mu_1}{\alpha \mu_1 + 1 - \alpha} = (1-\alpha)\phi_2$$

$$\frac{(1-\alpha)}{\alpha \mu_1 + 1 - \alpha} = (1-\alpha)\,\phi_2$$

$$\boxed{\frac{1}{\alpha \mu_1 + 1 - \alpha} = \phi_2}$$ ✓

(1) $\mu_0 = \gamma + \alpha\phi_0 + \alpha\phi_1\mu_0 + (1-\alpha)\phi_0$

$$\Downarrow$$

$$\boxed{\phi_0 = \frac{\mu_0(1-\alpha)}{1-\alpha+\alpha\mu_1} - \gamma}$$ ✓

See algebra on
next page

$$\mu_0 = \gamma + \alpha \phi_0 + \alpha \phi_1 \mu_0 + (1-\alpha)\phi_0$$

$$\mu_0 = \gamma + \alpha \phi_0 + \alpha \left(\frac{\mu_1}{\alpha \mu_1 + 1 - \alpha} \right) \mu_0 + (1-\alpha)\phi_0$$

$$\mu_0 = \gamma + \phi_0 + \alpha \left(\frac{\mu_1}{\alpha \mu_1 + 1 - \alpha} \right) \mu_0$$

$$\phi_0 = \mu_0 - \gamma - \alpha \left(\frac{\mu_1}{\alpha \mu_1 + 1 - \alpha} \right) \mu_0$$

$$\phi_0 = \frac{\mu_0(\alpha \mu_1 + 1 - \alpha) - \alpha \mu_1 \mu_0}{\alpha \mu_1 + 1 - \alpha} - \gamma$$

$$\phi_0 = \frac{\cancel{\mu_0 \alpha \mu_1} + \mu_0 - \alpha \mu_0 \cancel{- \alpha \mu_1 \mu_0}}{\alpha \mu_1 + 1 - \alpha} - \gamma$$

$$\boxed{\phi_0 = \frac{\mu_0(1-\alpha)}{1 - \alpha + \alpha \mu_1} - \gamma}$$

Notice the size of ϕ_2:

$$\phi_2 = \frac{1}{\alpha \mu_1 + 1 - \alpha} = \frac{1}{\alpha(\mu_1 - 1) + 1}$$

α is negative, and for "stable" money process $\mu_1 < 1$, so $\alpha(\mu_1 - 1)$ is positive

$$0 < \phi_2 < 1$$

Intuition?

Also,

$$\phi_3 = \frac{-1}{1-\alpha}$$

$1 - \alpha$ is positive and larger than 1

So

$$-1 < \phi_3 < 0$$

Intuition?