The Signal Extraction Problem

I. A Regression Analogy

A. Suppose that we have a sample of past values of x and y (expressed as deviations from means).

B. Now suppose that today we observe x_t, but our interest is in predicting y_t.

C. For the regression model:

$$y_t = bx_t + e_t$$

we would estimate b:

$$\hat{b} = \frac{\sum_t x_t y_t}{\sum_t x_t^2}$$

$$\hat{b} = \frac{\text{sample covariance }(x, y)}{\text{sample variance }(x)}$$

and we would forecast:

$$\hat{y}_t = \hat{b}x_t.$$

II. Signal Extraction

A. Now suppose that two random errors, e_t and u_t are to be drawn. These errors have the following properties:

$$E(e_t) = E(u_t) = E(e_t u_t) = 0$$
$$E(e_t^2) = s_e^2 \quad E(u_t^2) = s_u^2$$

B. Assume that the distributional information presented above is known.

C. Now suppose that e_t and u_t values are drawn, but only the sum, $s_t = e_t + u_t$, is observed.

D. I want to forecast u_t given my knowledge of s_t.

1. I know:

$$\text{Var}\,(s_t) = \text{Var}\,(e_t + u_t) = s_e^2 + s_u^2$$

$$\text{Cov}\,(s_t, u_t) = E\big[(e_t + u_t)u_t\big] = s_u^2$$

2. For a “regression” of u_t on s_t we have:

$$\hat{b} = \frac{\text{Cov}\,(s_t, u_t)}{\text{Var}\,(s_t)}$$

$$\hat{b} = \frac{s_u^2}{s_e^2 + s_u^2}$$

3. So we forecast:

$$u_t = \left(\frac{s_u^2}{s_e^2 + s_u^2}\right)s_t$$